

GROUPS QUASI-ISOMETRIC TO $\mathbf{H}^2 \times \mathbf{R}$

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In memory of Herbert Busemann.

INTRODUCTION

The most powerful geometric tools are those of differential geometry, but to apply such techniques to finitely generated groups seems hopeless at first glance since the natural metric on a finitely generated group is discrete. However Gromov recognized that a group can metrically resemble a manifold in such a way that geometric results about that manifold carry over to the group [18, 20]. This resemblance is formalized in the concept of a “quasi-isometry.” This paper contributes to an ongoing program to understand which groups are quasi-isometric to which simply connected, homogeneous, Riemannian manifolds [15, 18, 20] by proving that any group quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$ is a finite extension of a cocompact lattice in $Isom(\mathbf{H}^2 \times \mathbf{R})$ or $Isom(\widetilde{SL}(2, \mathbf{R}))$.

Theorem. *For any group Γ quasi-isometric to the hyperbolic plane cross the real line, there is an exact sequence*

$$0 \rightarrow A \rightarrow \Gamma \rightarrow G \rightarrow 0$$

where A is virtually infinite cyclic and G is a finite extension of a cocompact Fuchsian group.

With this result, the question of which finitely generated groups are quasi-isometric to each of Thurston’s eight geometries [33][31] remains open only for Sol. Tukia [35] for $n > 2$ and Gabai [14], or alternatively Casson–Jungreis [6], and Tukia [36] in dimension 2 show that any group quasi-isometric to \mathbf{H}^n can be realized as a finite extension of a discrete cocompact subgroup of the isometries of \mathbf{H}^n . The analogous result for $\mathbf{H}^2 \times \mathbf{R}$ is not true, because $\mathbf{H}^2 \times \mathbf{R}$ is quasi-isometric to $\widetilde{SL}(2, \mathbf{R})$, the universal cover of $SL(2, \mathbf{R})$. Gromov’s work on groups of polynomial growth [17] implies that any group quasi-isometric to \mathbf{R}^n is a finite extension of a discrete, cocompact subgroup of the isometry group of \mathbf{R}^n . Similarly for Nil. Kapovich and Leeb [22] show that quasi-isometries preserve the geometric decomposition of Haken manifolds. Their paper, which was written after the present paper, contains another proof, along quite different lines, of the main result in this paper. In their 1996 preprint [24], Kleiner and Leeb generalized the main results of the present paper to products of simply connected nilpotent Lie group with a symmetric space of non-compact type with no Euclidean de Rham factors. Chow [5] has shown that every group quasi-isometric to the complex hyperbolic plane is a finite extension of a discrete, cocompact subgroup of the isometry group of the complex hyperbolic plane. Pansu

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[27] has shown that in quaternionic and Cayley hyperbolic space every quasi-isometry is within bounded distance of an isometry, so that in fact every group quasi-isometric to one of these spaces is a finite extension of a group naturally isomorphic to a discrete cocompact subgroup of the isometry group. A series of papers by Pansu [27], Schwartz [29, 30], Farb-Schwartz [13], Kleiner-Leeb [23], Eskin-Farb [8], and Eskin [7] succeeded in classifying lattices in semi-simple Lie groups up to quasi-isometry. See [10] for a survey. Of course, the question may be asked for metric spaces other than symmetric spaces or even manifolds. Papers by Gromov and Stallings give the result for trees [19][32]. Farb and Mosher [11][12] prove quasi-isometric rigidity for the solvable Baumslag-Solitar groups.

The heart of the paper is contained in Sections 3 and 5. Section 1 gives some background and definitions and sets the notation for the paper. Section 2 describes a “quasi-action” of a group Γ quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$ on $\mathbf{H}^2 \times \mathbf{R}$. Section 3 uses geometric arguments to obtain an action of Γ on $\partial\mathbf{H}^2$ by quasimaps. Section 4 describes the application of results of Hinkkanen [21], Gabai [14], Casson–Jungreis [6], and Tukia [36] to show that this action is conjugate by a quasimetric map to an action by a Möbius group. Section 5 uses largely algebraic methods to show that the resulting action on \mathbf{H}^2 by isometries must have been properly discontinuous, and thus there is a map $\Phi : \Gamma \rightarrow G$ where G is a discrete, cocompact subgroup of the isometries of \mathbf{H}^2 . Section 6 uses geometric arguments from Section 3 to show that $\ker \Phi$ must be quasi-isometric to \mathbf{R} .

The aim of Section 3 is to show that the image of a horizontal hyperbolic plane in $\mathbf{H}^2 \times \mathbf{R}$ under any quasi-isometry induced by Γ must have sufficient horizontal expanse there is a natural map from $\partial\mathbf{H}^2$ to itself. The intuition behind the proof is that slicing $\mathbf{H}^2 \times \mathbf{R}$ vertically gives Euclidean planes, which have much less area than hyperbolic planes, so the image of a horizontal hyperbolic plane under a quasi-isometry of $\mathbf{H}^2 \times \mathbf{R}$ cannot be contained in a vertical slice. The proof formalizes this idea by showing that if the image of vertical geodesics, from a maximal family of geodesics all within a vertical cylinder all at least a certain distance apart, were to quasicross a vertical cross-section, then the number of quasicrossing points, all at least a certain distance apart, exceeds the number possible in a Euclidean disk.

Key to Section 5 is the notion of semilocal growth of a finitely generated subgroup of a Lie group. In particular, Theorem 5.19 states that any finitely generated, non-elementary, non-discrete subgroup of $PSL(2, \mathbf{R})$ has superpolynomial semilocal growth. The semilocal growth of a subgroup is strictly larger than its local growth, a notion due to Carrière [2]. In fact, all of the results of Section 5 can be obtained replacing semilocal growth with local growth, but the notion of semilocal growth seems more natural in this setting. It perhaps shows the importance of local growth and Theorem 5.19, which had been previously obtained by Carrière and Ghys [3], that they arose independently in different contexts.

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1. BACKGROUND AND DEFINITIONS

Two metric spaces are quasi-isometric if there is a relation between them which, except locally, does not increase or decrease distances too much and has the property that every point in each space is close to a point which is related to a point in the other space.

Definition 1.1. *A map $\psi : X \rightarrow Y$ is a $(\lambda, \epsilon, \delta)$ -quasi-isometry if it satisfies*

- (1) *(Lipschitz-in-the-large) $\frac{1}{\lambda}d(x_1, x_2) - \epsilon \leq d(\psi(x_1), \psi(x_2)) \leq \lambda d(x_1, x_2) + \epsilon$, and*
- (2) *(Almost surjectivity) For all $y \in Y$, there exists $y' \in Y$ with $d(y, y') \leq \delta$ such that, for some $x \in X$, $\psi(x) = y'$.*

There is some discrepancy in the literature as to whether quasi-isometries are required to be almost surjective or not. Throughout this paper, when we omit the δ in the definition of a quasi-isometry, we will mean that we are not requiring the almost surjectivity condition, or in other words, the map is a quasi-isometry in the sense given above only onto the range of the map.

Two metric spaces are quasi-isometric if there exist quasi-isometries between them that are almost inverses of each other. This idea is formalized in the following definition.

Definition 1.2. *Two metric spaces X and Y are $(\lambda, \epsilon, \delta)$ -quasi-isometric if there exist $(\lambda, \epsilon, \delta)$ -quasi-isometries $\psi : X \rightarrow Y$ and $\omega : Y \rightarrow X$ such that for some κ*

- (1) *$d(x, \omega(\psi(x))) \leq \kappa$ for all $x \in X$, and*
- (2) *$d(y, \psi(\omega(y))) \leq \kappa$ for all $y \in Y$.*

Any group Γ with generating set S , where S contains the inverse of any element in S , has a natural metric, the word metric d_S , given by

$$d_S(g, h) = \min\{n \mid ga_1a_2 \dots a_n = h \text{ with } a_i \in S\}.$$

Word metrics coming from two different finite generating sets S and S' are equivalent in the sense that there is a constant λ such that

$$\frac{1}{\lambda}d_S(x, y) \leq d_{S'}(x, y) \leq \lambda d_S(x, y).$$

So up to quasi-isometry there is a natural metric on a finitely generated group.

Note 1.3. *Whenever we say “a group is quasi-isometric” implicitly we will be talking about a finitely generated group endowed with one of the equivalent word metrics.*

2. A QUASIACTION OF Γ ON $\mathbf{H}^2 \times \mathbf{R}$

Throughout this paper Γ will denote a group $(\lambda, \epsilon, \delta)$ -quasi-isometric to X via the word metric on Γ coming from a finite generating set.

Definition 2.1. A $(\lambda_0, \epsilon_0, \delta_0, \kappa)$ -quasi-action of a group Γ on a metric space X consists of a family of $(\lambda_0, \epsilon_0, \delta_0)$ -quasi-isometries $\phi_u : X \rightarrow X$ such that $d(\phi_u \phi_{u'}(x), \phi_{uu'}(x)) \leq \kappa$ for all $u, u' \in \Gamma$ and all $x \in X$.

Note that were κ zero, we would have an action of Γ on X . Also it's worth pointing out that the ϕ_u 's are not necessarily homeomorphisms.

Observation 2.2. A group Γ quasi-isometric to a $\mathbf{H}^2 \times \mathbf{R}$ has a natural quasi-action on $\mathbf{H}^2 \times \mathbf{R}$, with $\kappa = \lambda\delta + \epsilon$, given by $\phi_u = \psi \circ u \circ \omega$, where ψ and ω are the $(\lambda, \epsilon, \delta)$ -quasi-isometries from the definition of $\mathbf{H}^2 \times \mathbf{R}$ and Γ being quasi-isometric, and $u \in \Gamma$ acts on Γ by left multiplication.

3. CONSTRUCTING AN ACTION OF Γ ON $\partial\mathbf{H}^2$.

Throughout this section “vertical” will refer to the \mathbf{R} coordinates and “horizontal” to the \mathbf{H}^2 coordinates. Let π be the projection of $\mathbf{H}^2 \times \mathbf{R}$ onto $\mathbf{H}^2 \times \{0\}$. In order to obtain an action of Γ on $\partial\mathbf{H}^2$ of \mathbf{H}^2 we want to show that each $\pi \circ \phi_u$ restricted to the horizontal plane $\mathbf{H}^2 \times \{0\}$ is a quasi-isometry. The idea is to show that each ϕ_u must preserve horizontal and vertical in some rough sense. The proof exploits the fact that there is a lot more room in a disk in the hyperbolic plane than in a disk in the Euclidean plane. We begin with two estimates exhibiting this difference.

Estimate 3.1. An upper bound for the number of disks of radius r in a disk of radius R all at least $2s$ apart in Euclidean space is given by

$$\left(\frac{(R+s)}{(r+s)} \right)^2.$$

Recall that the area of a hyperbolic disk of radius R is given by $2\pi(\cosh R - 1)$.

Estimate 3.2. A lower bound for a maximal number of disks of radius r in a disk of radius R all at least $2s$ apart in hyperbolic space is given by

$$\frac{e^R - 2}{2(\cosh(2(r+s)) - 1)}.$$

By “the horizontal distance between points x and y ,” I will mean the distance in $\mathbf{H}^2 \times \{0\}$ between $\pi(x)$ and $\pi(y)$, and by “vertical distance,” the distance between the \mathbf{R} -coordinates.

Proposition 3.3. Let $\phi : \mathbf{H}^2 \times \mathbf{R} \rightarrow \mathbf{H}^2 \times \mathbf{R}$ be a (λ, ϵ) -quasi-isometry. Let C_0 be a vertical solid cylinder of radius r , and let p_0 and q_0 be points in C_0 such that $\phi(p_0)$ and

$\phi(q_0)$ have vertical distance no greater than h_0 . Then p_0 and q_0 must be no farther apart than

$$ch_0 + c$$

where c is a constant depending only on λ , ϵ and r .

Proof. Let $d(p_0, q_0) = L$. Let w denote the horizontal distance between $\phi(p_0)$ and $\phi(q_0)$ and h the vertical. Now

$$\begin{aligned} L/\lambda - \epsilon &\leq d(\phi(p_0), \phi(q_0)) \\ &= \sqrt{w^2 + h^2} \\ &\leq w + h_0. \end{aligned}$$

So

$$w \geq L/\lambda - \epsilon - h_0.$$

Project $\phi(p_0)$ and $\phi(q_0)$ onto $\mathbf{H}^2 \times \{0\}$. Let α be the geodesic which intersects the geodesic between the projected images of $\phi(p_0)$ and $\phi(q_0)$ perpendicularly at the point halfway between them.

For every vertical solid cylinder C of radius r whose central axis is distance D from the central axis of C_0 , there are points p and q that are the translation of p_0 and q_0 under the translation that takes the central axis of C_0 to the central axis of C and preserves the vertical height of each point. Let γ be the geodesic between p and q . The points $\phi(p)$ and $\phi(q)$ are on opposite sides of $\{\alpha\} \times \mathbf{R}$, if

$$d(\phi(p_0), \phi(p)) \leq 1/2w$$

and

$$d(\phi(q_0), \phi(q)) \leq 1/2w.$$

Let R be the maximal distance D for which these inequalities hold:

$$R = \frac{1}{2\lambda}(L/\lambda - 3\epsilon - h_0).$$

Let \mathcal{C} be a maximal set of cylinders of radius r all s apart within the cylinder of radius R about C_0 , where $s = \lambda(1 + 2\epsilon)$. For each cylinder C in \mathcal{C} , let x be a point in $\{\alpha\} \times \mathbf{R}$ that is as close as possible to $\phi(\gamma)$. Such points are called quasicrossing-points of $\phi(\gamma)$ with $\{\alpha\} \times \mathbf{R}$ since $d(x, \phi(\gamma)) \leq \epsilon/2$. Two quasicrossing-points x and x' associated with cylinders C and C' in \mathcal{C} must be at least 1 apart as

$$\begin{aligned} d(x, x') &\geq d(y, y') - d(x, y) - d(x', y') \\ &\geq s/\lambda - \epsilon - \epsilon/2 - \epsilon/2 = 1, \end{aligned}$$

where y and y' are points on $\phi(\gamma)$ and $\phi(\gamma')$ within $\epsilon/2$ of x and x' respectively.

To see that

$$d(x, x') \leq \lambda(L + R) + 2\epsilon,$$

let z and z' be points on γ such that $\phi(z) = y$ and $\phi(z') = y'$. Since all points on γ and γ' lie within a cylinder of radius R and height L , $d(z, z') \leq L + R$. So

$$\begin{aligned} d(x, x') &\leq d(y, y') + d(x, y) + d(x', y') \\ &\leq \lambda d(z, z') + \epsilon + \epsilon \\ &\leq \lambda(L + R) + 2\epsilon. \end{aligned}$$

By Estimate 3.2 there are at least $\frac{e^R - 2}{2(\cosh(2(r+s/2)) - 1)}$ vertical cylinders of radius r in \mathcal{C} . Let X be a set of quasicrossing-points, one for each cylinder C in \mathcal{C} . By estimate 1 there are at most

$$(2\lambda(L + R) + 4\epsilon + 1)^2$$

such points. Thus

$$\frac{e^R - 2}{2(\cosh(2(r + s/2)) - 1)} \leq (2\lambda(L + R) + 4\epsilon + 1)^2.$$

Solving for L in the definition of R gives

$$L = 2\lambda^2 R + 3\lambda\epsilon + \lambda h_0.$$

Substituting this expression for L in the above inequality shows that e^R must be at most some quadratic function of λ , ϵ , r and h_0 . Furthermore

$$e^R \leq b_2 R^2 + b_1 R + b_0,$$

where the coefficients b_0 , b_1 and b_2 are polynomial functions of h_0 of degree no more than 2. The first four terms of the Taylor expansion of e^R tell us that R must be no more than $b_2 + 2b_1 + 6b_0$. So R is no more than some quadratic function of h_0 . The definition of R together with this bound give a similar bound on L ,

$$L \leq c_2 h_0^2 + c_1 h_0 + c_0,$$

where the coefficients depend only on λ , ϵ , and r . We can get a linear bound on L with respect to h_0 as follows. Let $c = c_2 + c_1 + c_0$. A geodesic between $\phi(p_0)$ and $\phi(q_0)$ of vertical distance $h \leq h_0$ can be split up into no more than $h_0 + 1$ pieces of vertical distance no greater than 1. Using the above bound on each piece and adding them together, we know that p_0 and q_0 can be no farther apart than $c(h_0 + 1)$. \square

Remark 3.4. For future reference note that $c = aS$ where $S = 2(\cosh(2r + \lambda(1 + 2\epsilon)) - 1)$ and a is some constant depending only on λ and ϵ .

Corollary 3.5. *Let $\phi_u : \mathbf{H}^2 \times \mathbf{R} \rightarrow \mathbf{H}^2 \times \mathbf{R}$ be a (λ, ϵ) -quasi-isometry coming from the quasi-action of Γ on $\mathbf{H}^2 \times \mathbf{R}$. Let x and y be points D apart in some horizontal \mathbf{H}^2 . Then the horizontal distance l between $\phi_u(x)$ and $\phi_u(y)$ must be at least*

$$l \geq \ln\left(\left(\frac{\frac{D}{\lambda} - \epsilon}{a(2\kappa + 2\delta + 1)} - 1\right) - \lambda(2\epsilon + 1) + 1\right)$$

where κ is the quasi-action constant of Observation 2.2, and a depends only on λ and ϵ .

Proof. Since $d(\phi_{u^{-1}}\phi_u(x), x) \leq d(\phi_{u^{-1}}\phi_u(x), \phi_e(x)) + d(\phi_e(x), x) \leq \kappa + \delta$ and similarly for y , the vertical distance between $\phi_{u^{-1}}(\phi_u(x))$ and $\phi_{u^{-1}}(\phi_u(y))$ must be less than $2(\kappa + \delta)$. Hence by Proposition 3.3, $d(\phi_u(x), \phi_u(y)) \leq 2c(\kappa + \delta) + c$. By Remark 3.4,

$$d(\phi_u(x), \phi_u(y)) \leq 2a \cosh(2r + \lambda(2\epsilon + 1)) - 1)(2(\kappa + \delta) + 1),$$

where r is the radius of a vertical cylinder containing $\phi_u(x)$ and $\phi_u(y)$, so can be taken to be half the horizontal distance l between $\phi_u(x)$ and $\phi_u(y)$. It is l we are trying to bound from below. So

$$\frac{D}{\lambda} - \epsilon \leq d(\phi_u(x), \phi_u(y)) \leq (2a \cosh(l + \lambda(2\epsilon + 1)) - 1)(2(\kappa + \delta) + 1),$$

from which the result follows. \square

Proposition 3.6. $\pi \circ \phi : \mathbf{H}^2 \times \{0\} \rightarrow \mathbf{H}^2 \times \{0\}$ is a (λ', ϵ') -quasi-isometry where $\lambda' = \max\{\lambda, D'\}$ and $\epsilon' = \max\{\epsilon, 1\}$ taking $D' = \lambda a(2\kappa + 2\delta + 1)(e^{\lambda(2\epsilon + 1)} + 1) + \epsilon$.

Proof. The projection π is distance nonincreasing, so for any two points x and y in $\mathbf{H}^2 \times \{0\}$ we know

$$d(\pi \circ \phi(x), \pi \circ \phi(y)) \leq \lambda d(x, y) + \epsilon.$$

We need to show that $\pi \circ \phi$ does not decrease distances too much. By Corollary 3.5 the images under $\pi \circ \phi$ of any two points at least distance D' apart cannot be closer than 1 unit apart. Let $L = d(\pi \circ \phi(x), \pi \circ \phi(y))$, and let γ be the geodesic between $\pi \circ \phi(x)$ and $\pi \circ \phi(y)$. Let the x_i be $N \leq L + 1$ points along γ such that successive x_i are within distance 1 of each other and $x_0 = \pi \circ \phi(x)$ and $x_N = \pi \circ \phi(y)$. So successive quasi-preimages of the x_i must be no more than D' apart. Hence

$$d(x, y) \leq D'(L + 1) = D'd(\pi \circ \phi(x), \pi \circ \phi(y)) + D'.$$

Therefore,

$$d(\pi \circ \phi(x), \pi \circ \phi(y)) \geq \frac{1}{D'}d(x, y) - 1.$$

\square

A quasisymmetric map of S^1 viewed as the boundary of the Poincaré model for the hyperbolic plane is a map that extends to a quasiconformal map of \mathbf{H}^2 . If we view \mathbf{H}^2

as the upper half plane of \mathbf{C} , Beurling and Ahlfors [1] showed that f is quasiconformal and fixes the point at infinity exactly when there exists a constant ϵ such that

$$1/\epsilon \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \epsilon.$$

It is well-known (see for example [9][16][26]) that any quasi-isometry of \mathbf{H}^2 extends to a map on $\partial\mathbf{H}^2$ that is quasiconformal and the quasiconformality constant only depends on the quasi-isometry constants. The central idea is that the image of any geodesic under a quasi-isometry lies in the B neighborhood of a geodesic, where the constant B depends only on λ and ϵ . Although we only had a quasi-action of Γ on $\mathbf{H}^2 \times \mathbf{R}$ and also on $\mathbf{H}^2 \times \{0\}$, we get a true action on $\partial\mathbf{H}^2$ for the following reasons. Since the images of a geodesic in $\mathbf{H}^2 \times \{0\}$ under $\pi \circ \phi_{uu'}$ and $(\pi \circ \phi_u) \circ (\pi \circ \phi_{u'})$ are within bounded distance of each other, they must be within bounded distance of the same geodesic. To see where a point x in $\partial\mathbf{H}^2$ goes under a quasi-isometry ϕ , take a geodesic ray $\gamma(t)$ with x as its endpoint. The subset $\phi(\gamma)$ is within a bounded distance of some geodesic ray. Map x to the appropriate endpoint of this geodesic ray. Since this geodesic ray is the same for both maps, both maps must give the same action on the boundary.

Corollary 3.7. *There is a canonical surjective homomorphism $\Xi : \Gamma \rightarrow F$ where F is the uniformly quasiconformal group consisting of the boundary values of the $\pi \circ \phi_u$'s.*

4. THE ACTION OF Γ ON $\partial\mathbf{H}^2$ IS CONJUGATE BY A QUASICONFORMAL MAP TO AN ACTION BY A MÖBIUS GROUP.

Let F^+ be the group of orientation preserving elements of F . This section shows that F^+ may be conjugated by a quasiconformal map to a Möbius group G . The results of this section were previously known. For example, Lemma 4.3 is a special case of Theorem 9 in [25], and Lemma 4.4 follows trivially from results in Section C of [35].

For F^+ not discrete in the topology of pointwise convergence (a case we will rule out in section 5), we may use the following theorem of Hinkkanen.

Theorem 4.1(Hinkkanen) [21]. *Let G be a uniformly quasiconformal group containing a sequence of distinct elements that tend to the identity pointwise. Then G is a quasiconformal conjugate of a Möbius group.*

Throughout the rest of the section we will assume F^+ is discrete. Any discrete uniformly quasiconformal group of orientation preserving homeomorphisms is a convergence group, so the following theorem holds for F^+ .

Theorem 4.2(Casson–Jungreis [6], Gabai [14], Tukia [36]). *G is a convergence group if and only if G is conjugate in $\text{Homeo}(S^1)$ to the restriction of a Fuchsian group.*

This theorem was proved by Gabai [14], and independently by Casson–Jungreis [6], building on work of Tukia [36]. The rest of this section is devoted to showing that the map conjugating F^+ to a Möbius group may be taken to be quasiconformal.

Lemma 4.3. *Say $\Theta : \Gamma \rightarrow \mathbf{H}^2$ and $\Psi : \Gamma \rightarrow \mathbf{H}^2$ are two maps which induce a quasi-isometry between Γ and \mathbf{H}^2 . Then the quasisymmetric maps they induce on $\partial\mathbf{H}^2$ are conjugate by a quasisymmetric map.*

Proof. Define $H : \Theta(\Gamma) \rightarrow \Psi(\Gamma)$ to be the composition of Ψ with a quasi-inverse of Θ . As we saw in the previous section H can be extended to a quasi-isometry of \mathbf{H}^2 to itself, which induces a quasisymmetric map h on $\partial\mathbf{H}^2$. By examining the construction, we see that h conjugates the map induced on $\partial\mathbf{H}^2$ by Ψ to the one induced by Θ . \square

Lemma 4.4. *For any (λ, ϵ) -quasi-isometry $f : \mathbf{H}^2 \rightarrow \mathbf{H}^2$ that has the same boundary values as some isometry $g : \mathbf{H}^2 \rightarrow \mathbf{H}^2$, there is a constant L such that $d(f(x), g(x)) \leq L$ for all $x \in \mathbf{H}^2$, where L is dependent only on λ and ϵ .*

Proof. Let y_0, y_1, y_2 be vertices of an ideal triangle such that x is within the same distance, say P , of each of the geodesics connecting the three points. Denote by Y_i the geodesic connecting y_i and $y_{(i+1) \bmod 3}$. As was explained in the paragraph following the proof of Lemma 3.6, $f(Y_i)$ is a curve that remains within distance D (depending only on λ and ϵ) of $g(Y_i)$. The point $f(x)$ is within distance $\lambda P + \epsilon$ of each $f(Y_i)$, so must be within $\lambda P + \epsilon + D$ of each $g(Y_i)$. The intersection of these regions has bounded diameter, say L . Since $g(x)$ is certainly within this region, $f(x)$ must be within L of $g(x)$. \square

Lemma 4.5. *Any finitely generated, discrete, uniformly (λ, ϵ) -quasisymmetric group G acting on S^1 , that induces a cocompact action on the space of triples T , is quasi-isometric to \mathbf{H}^2 .*

Proof. To any ordered triple of distinct points in S^1 we associate a point in hyperbolic space as follows. Connect the first two points by a geodesic and drop a perpendicular from the third. The intersection will be the point associated to the triple. Choose a triple t_0 whose associated point is x_0 . Let $f : G \rightarrow \mathbf{H}^2$ be the map sending an element $a \in G$ to the point associated to the triple $a(t_0)$. We will show that f is a quasi-isometry. For any element $a \in G$, choose a (λ, ϵ) -quasi-isometry $\eta_a : \mathbf{H}^2 \rightarrow \mathbf{H}^2$ with boundary values a . Note $\eta_a(f(a'))$ must be within $2B$ of $f(aa')$ (where B was defined in the paragraph following Proposition 3.6), since $\eta_a(f(a'))$ must be within B of each of the geodesics used to construct $f(aa')$.

Since G is cocompact on the space of triples, there is a constant E such that every point in \mathbf{H}^2 is within E of some point in the image of f . Let J' be a finite generating set for G . Enlarge J' to J by including all elements a such that $d(\eta_a(x_0), x_0) \leq 3\lambda E + \epsilon + 6B$. By the discreteness of G , there are only finitely many such elements. Let $M = \max_{b \in J} d(f(b), x_0)$. We are now ready to check that f is a quasi-isometry. If

$d(a, a') = m$, then for some $b_i \in J$, $a = a'b_1b_2 \dots b_m$. So,

$$\begin{aligned}
d(f(a), f(a')) &\leq d(f(a'b_1 \dots b_m), f(a'b_1 \dots b_{m-1})) \\
&\quad + d(f(a'b_1 \dots b_{m-1}), f(a'b_1 \dots b_{m-2})) + \dots + d(f(a'b_1), f(a')) \\
&\leq d(\eta_{a'b_1 \dots b_{m-1}} f(b_m), \eta_{a'b_1 \dots b_{m-1}} f(e)) + \dots + d(\eta_{a'} f(b_1), \eta_{a'} f(e)) + 4mB \\
&\leq \lambda d(f(b_m), f(e)) + \epsilon + \dots + \lambda d(f(b_1), f(e)) + \epsilon + 4mB \\
&\leq (\lambda M + \epsilon + 4B)m \\
&= (\lambda M + \epsilon + 4B)d(a, a').
\end{aligned}$$

To get the lower bound, we compute as follows. Let $d(f(a), f(a')) = L$. Divide the geodesic between $f(a)$ and $f(a')$ into $[L/E] + 1$ segments of length no more than E , with endpoints x_0, x_1, \dots, x_N , where $f(a) = x_0$ and $f(a') = x_N$. Each x_i is within E of some point $f(a_i)$. By construction, for every i , $d(f(a_{i-1}), f(a_i)) \leq 3E$. We wish to show that there is a generator taking a_i to a_{i+1} for all i . Recall that $d(\eta_a(f(a'), f(aa'))) \leq 2B$ and $x_0 = f(e)$. Note that

$$\begin{aligned}
d(x_0, f(a_{i-1}^{-1}a_i)) &\leq d(x_0, \eta_{a_{i-1}^{-1}}(f(a_{i-1}))) + d(\eta_{a_{i-1}^{-1}}(f(a_{i-1})), \eta_{a_{i-1}^{-1}}(f(a_i))) \\
&\quad + d(\eta_{a_{i-1}^{-1}}(f(a_i)), f(a_{i-1}^{-1}a_i)) \\
&\leq 2B + 3\lambda E + \epsilon + 2B.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
d(\eta_{a_{i-1}^{-1}a_i}(x_0), x_0) &\leq d(x_0, f(a_{i-1}^{-1}a_i)) + d(f(a_{i-1}^{-1}a_i), \eta_{a_{i-1}^{-1}a_i}(x_0)) \\
&\leq 3\lambda E + \epsilon + 6B.
\end{aligned}$$

So for all i , $a_{i-1}^{-1}a_i = b_i$ is in the generating set for G .

Since

$$a' = a_N = a_{N-1}b_N = \dots = ab_1b_2 \dots b_N,$$

we have

$$\begin{aligned}
d(a, a') &\leq N \\
&= [L/E] + 1 \\
&\leq \frac{1}{E}d(f(a), f(a')) + 1.
\end{aligned}$$

□

Observation 4.6. *Section 2 together with the last paragraph of section 3 shows how a quasi-isometry of a group G with the hyperbolic plane induces an action on $\partial \mathbf{H}^2$. It is easy to check that the action of F^+ on $\partial \mathbf{H}^2$ that we get from the quasi-isometry of F^+ with \mathbf{H}^2 given by the Lemma 4.5 is the same action of F^+ on $\partial \mathbf{H}^2$ that we started with.*

Proposition 4.7. *F^+ is conjugate to a Möbius group by a quasisymmetric homeomorphism $h : S^1 \rightarrow S^1$.*

Proof. For F^+ not discrete this is a result of Hinkkanen [21].

Gabai [G], or Casson–Jungreis [6], together with Tukia [36] show that any discrete convergence group (in particular, any quasisymmetric group) is conjugate to a finite extension of a Fuchsian group by some homeomorphism h_1 of S^1 . Our F^+ acts cocompactly on the space of triples, so $G^+ = h_1 F^+ h_1^{-1}$ must also. Thus G^+ is a Fuchsian group acting discretely and cocompactly on the space of triples, so when we extend the action to \mathbf{H}^2 we get a discrete cocompact group of hyperbolic isometries. This gives us a quasi-isometry of G^+ with \mathbf{H}^2 . We get a different quasi-isometry $\Psi : G \rightarrow \Psi(G)$ by using the isometry of G^+ with F^+ and applying Lemma 4.5 to F^+ . Applying Lemma 4.3 to these two quasi-isometries, which have the same boundary values by Observation 4.6, yields the desired result.

□

Let $\Phi : \Gamma \rightarrow G$ be the homomorphism we have obtained where G is either the Möbius group G^+ or a $\mathbf{Z}/2\mathbf{Z}$ extension of G^+ , depending on whether F contained orientation reversing elements or not.

5. THE DISCRETENESS OF THE IMAGE OF Φ

Our goal now is to show that G is discrete. The idea is that any non-elementary Möbius group which is not discrete has many more small elements than G . More precisely, choose a left-invariant metric on $PSL(2, \mathbf{R})$. For any $\epsilon > 0$, denote by N_ϵ the ϵ -neighborhood of the identity in $PSL(2, \mathbf{R})$. For any finitely generated subgroup H of $PSL(2, \mathbf{R})$, let H_ϵ^n denote the set of $h_u \in N_\epsilon$ such that u has word length less than or equal to n . Our claim is that for some ϵ (specified later), $|G_\epsilon^n|$, the number of elements in G_ϵ^n , grows linearly with n , while for any finitely generated nonelementary group H that is not discrete $|H_\epsilon^n|$ grows exponentially in n .

Definition 5.1. *Let G be a finitely generated group with word metric d imbedded in another group L with metric ρ . Then the semilocal growth of G in L is defined to be the growth rate of the number of elements in*

$$G_\epsilon^n = \{g \in G \mid d(g, e) \leq n, \rho(g, e) \leq \epsilon\}.$$

Note 5.2. *This notion of local growth is strictly bigger than Carrière’s notion of local growth [2], which counts only elements g such that the subwords of increasing length making up g are all in the ϵ -neighborhood of the identity. More, precisely it counts only $g = a_{i_1} a_{i_2} \dots a_{i_m}$ where $g_j = a_{i_1} a_{i_2} \dots a_{i_j} \in N_\epsilon$ for all $j \leq m$, where the a_k are generators for G . All of the results hold equally well for local growth as for semilocal growth, but as local growth imposed an extra, unnecessary condition on the elements we are counting, we will prove the results for semilocal growth.*

G has linear semilocal growth in $PSL(2, \mathbf{R})$.

Recall that $\psi : \Gamma \rightarrow \mathbf{H}^2 \times \mathbf{R}$ is the quasi-isometry that came from the definition of Γ and $\mathbf{H}^2 \times \mathbf{R}$ being quasi-isometric.

Observation 5.3. *The number of $u \in \Gamma$ such that the images of $z_0 = \psi(e)$ under ϕ_u lies in a vertical cylinder K of height 1 and radius R centered about the vertical geodesic through z_0 is bounded by some constant N depending only on λ , ϵ and R .*

Observation 5.4. *The number of $u \in \Gamma$ of word length n such that $\phi_u(z_0)$ lies in C_R , the vertical cylinder of radius R centered about z_0 , is bounded by $An + B$ where $A = 2\lambda N$ and $B = (2\epsilon + 1)N$.*

Theorem 5.5. *For any ϵ , $|G_\epsilon^n|$ grows linearly with n . More explicitly, there exist constants A and B such that the number of elements in G_ϵ^n is less than or equal to $An + B$.*

Proof. For any $u \in \Gamma$ let $g_u = \Phi(u)$. There is some constant r_ϵ such that any g_u within ϵ of the identity moves x_0 no more than r_ϵ . Let f_u be the map whose conjugate under h is g_u . By the Lemma 4.4, f_u must not move x_0 by more than $L + r_\epsilon$. The map f_u came from the projection of the image of the horizontal \mathbf{H}^2 containing $\psi(e)$ under ϕ_u . Since $\psi(e)$ projects to x_0 , we conclude that ϕ_u must send $\psi(e)$ to some point in the vertical cylinder of radius $L + r_\epsilon$ centered about $\psi(e)$. Taking $R = L + r_\epsilon$ in Observation 5.4, the number of such u of word length less than or equal to n is bounded by $An + B$. \square

Lemma 5.6. *The kernel of Φ is either finite or contains an element of infinite order.*

Proof. Passing if necessary to an index 2 subgroup, we may assume that all elements u of $\ker \Phi$ are end-preserving in the sense that the image under ϕ_u of a sequence of points whose \mathbf{R} components tend to $+\infty$ (resp. $-\infty$) also have \mathbf{R} components which tend to $+\infty$ (resp. $-\infty$). Our goal will be to show that if an element $u \in \ker \Phi$ has finite order, then $\psi(u)$ and $\psi(e)$ are within a vertical distance of $\lambda c + \epsilon$ of each other, where c is the constant of Proposition 3.3. This would imply that there are only finitely many elements of finite order, since the image under Ψ of any element of finite order in the kernel would have to be in the vertical cylinder C_0 of radius L centered at z_0 bounded above and below by $\lambda(c\epsilon/2 + c)\epsilon + 2(\lambda\delta + \epsilon)$. But by Observation 5.3, the number of such points is finite.

Say u has finite order n and the vertical distance between $\psi(u)$ and $\psi(e)$ is greater than $\lambda c + \epsilon$. We assume without loss of generality that $\psi(u)$ is above $\psi(e)$. Then for some k , $\psi(u^k)$ is above $\psi(u^{k+1})$. Draw a geodesic segment between these two points and then continue it vertically upward from $\psi(u^k)$ and vertically downward from $\psi(u^{k+1})$. Call this curve γ . Under $\phi_{u^{n-k}}$, $\psi(u^k)$ is sent to $\psi(e)$ and $\psi(u^{k+1})$ is sent to $\psi(u)$. Since we are assuming ϕ_u is end-preserving, it follows that there is some point p on γ above $\psi(u^k)$, such that $\phi_u(p)$ is within $\epsilon/2$ of being at the same height as $\psi(u)$. From the previous section we know that this means that p and $\psi(u^{k+1})$ must be no farther than $c\epsilon/2 + c$ apart vertically. Thus,

$$d(\psi(u^{k+1}), \psi(u^k)) \leq d(\psi(u^{k+1}), p) \leq c\epsilon/2 + c.$$

Therefore,

$$\begin{aligned} d(\psi(e), \psi(u)) &\leq d(\phi_{u^{n-k}}\psi(u^{k+1}), \phi_{u^{n-k}}\psi(u^k)) + 2(\lambda\delta + \epsilon) \\ &\leq \lambda d(\psi(u^{k+1}), \psi(u^k)) + \epsilon + 2(\lambda\delta + \epsilon) \\ &\leq \lambda(c\epsilon/2 + c)\epsilon + 2(\lambda\delta + \epsilon). \end{aligned}$$

□

Theorem 5.7. *The kernel is either finite or quasi-isometric to \mathbf{R} .*

Proof. Suppose the kernel is infinite. Let $u \in \ker \Phi$ be an element of infinite order, whose existence is guaranteed by the previous Lemma. We may assume without loss of generality that ϕ_u is end-preserving and that u was in our generating set for Γ to begin with. Since u is of infinite order, there are infinitely many $\psi(u^i)$. Since all $\psi(u^i)$ are in C_0 , the vertical distance between $\psi(u^n)$ and z_0 must be greater than c , for n sufficiently large. Let $u^n = v$. Without loss of generality we may assume that $\psi(v)$ is above z_0 . By an argument similar to that of the previous Lemma, $\psi(v^{k+1})$ is above $\psi(v^k)$. Let $d(\psi(e), \psi(v)) = h$. Then for all k , $d(\psi(v^{k+1}), \psi(v^k)) \leq \lambda h + \epsilon$. Thus every point in C_0 is within $\frac{1}{2}(\lambda h + \epsilon) + L$ of one of the $\psi(v^k)$.

Say $w \in \ker \Phi$. Then $\psi(w)$ is within $\frac{1}{2}(\lambda h + \epsilon) + L$ of some $\psi(v^k)$. Thus,

$$\begin{aligned} d(z_0, \psi(v^{-k}w)) &\leq \lambda d(\phi_{v^k}(z_0), \phi_{v^k}(\psi(v^{-k}w))) + \epsilon \\ &= \lambda d(\psi(v^k), \psi(w)) + \epsilon \\ &\leq \lambda\left(\frac{1}{2}(\lambda h + \epsilon) + L\right) + \epsilon. \end{aligned}$$

There are only finitely many such points so $\langle v \rangle$ is of finite index in $\ker \Phi$. Thus $\ker \Phi$ is quasi-isometric to \mathbf{Z} which in turn is quasi-isometric to \mathbf{R} . □

Lemma 5.8. *The group G is a non-elementary subgroup of $PSL(2, \mathbf{R})$.*

Proof. By construction G does not fix a point or preserve an axis in \mathbf{H}^2 . Say G fixes a point on $\partial\mathbf{H}^2$. Then G would be solvable. Since $\ker \Phi$ is either finite or a finite extension of \mathbf{Z} , G solvable would imply that Γ was amenable. But Γ cannot be amenable since it is quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$. □

Theorem 5.9. *The group G is a non-elementary subgroup of $PSL(2, \mathbf{R})$ with the property that $|G_\epsilon^n|$ grows linearly.*

Non-discrete, finitely generated, non-elementary subgroups of $PSL(2, \mathbf{R})$ have exponential semilocal growth.

In order to show that $|H_\epsilon^n|$ grows quickly we need a tool for constructing small elements and another to make sure we can construct enough. The Zassenhaus Lemma will perform the first task while Tits's Theorem will do the second. We need the following Lemmas before we can apply the results.

Lemma 5.10. *G is either discrete or its closure is all of $PSL(2, \mathbf{R})$.*

Proof. By construction G does not fix any point in \mathbf{H}^2 or its boundary, and also does not preserve any axis. It is a well-known fact (see for example [4], Theorem 4.4.7) that the only closed subgroups of $PSL(2, \mathbf{R})$ either fix one of the above, are discrete, or are all of $PSL(2, \mathbf{R})$. \square

Let H be a subgroup of $PSL(2, \mathbf{R})$ that is not discrete and whose closure is all of $PSL(2, \mathbf{R})$.

Lemma 5.11. *H is not virtually solvable.*

Proof. The Lemma follows from the fact that a virtually solvable subgroup has virtually solvable closure. \square

Lemma 5.12. (Zassenhaus [37]) *There is a constant ϵ_0 such that for any $r \leq \epsilon_0$, if f and g are within distance r from the identity, then $[f, g]$ is also.*

A proof of this Lemma may be found in Raghunathan's book [28].

We will use this Lemma to construct more small elements from a few small elements. But we need a way to check that we are indeed constructing new and different elements by conjugating. Tits's alternative says that any subgroup of a linear group is either virtually solvable or contains a free group on two generators. We need a little more; we need a free group generated by two small generators in the Zassenhaus sense.

Note 5.13. *Unknown to me at the time, Carrière and Ghys [3] had previously proved the existence of such a free group along similar lines.*

We will construct these elements using the following fact.

Proposition 5.14. *Any finitely generated nonelementary subgroup H of the group $PSL(2, \mathbf{R})$ that is not discrete must contain an elliptic element of infinite order.*

Proof. By Selberg's Lemma, H contains a finite index normal subgroup N containing no non-trivial elements of finite order, from which it follows that H has no finite order elements of order greater than the index of N in H . But H is dense in $PSL(2, \mathbf{R})$, so there must be elements of H which get close to finite order elliptics of higher orders. The only possibilities are elliptics of infinite order, so H must contain at least one. \square

Let α be an infinite order elliptic element lying in H whose existence is guaranteed by the previous Proposition. The eigenvalues of α all have norm 1, but are not roots of unity, so they have infinite order in k^* , the subfield of \mathbf{C} generated by the matrix entries and eigenvalues of the generators for H . Thus we may use the following Lemma of Tits [32, 4.1], taking t to be one of α 's eigenvalues.

Lemma 5.15. *Let k be a finitely generated field and let $t \in k$ be an element of infinite order. Then there exists a locally compact field k' endowed with an absolute value ω and a homomorphism $\sigma : k \rightarrow k'$ such that $\omega(\sigma(t)) \neq 1$.*

Tits finds subsets S of H and \mathcal{U} of $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ with the property that for any $s_1, s_2 \in S$ with $s_1 \neq s_2$ and $(s_1, s_2) \in \mathcal{U}$ there exists a positive power m such that s_1^m and s_2^m generate a free group. He constructs these sets as follows.

The set S consists of elements $s \in H$ with eigenvalues λ_1, λ_2 such that $\omega(\sigma(\lambda_i)) \neq 1$ for $i = 1, 2$. Let \mathcal{U} be the set of $(x, y) \in SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ such that x and y are semisimple with distinct eigenvalues and that, for any eigenvectors v and w of x and y respectively, $w^*(v) \neq 0$.

Our infinite order elliptic α clearly lies in S . Since the closure of H is all of $PSL(2, \mathbf{R})$, H must contain a hyperbolic element. Conjugate α by this hyperbolic element to get an elliptic β with a different fixed point. β is also semi-simple and has the same eigenvalues as α , so β is also in S . A straightforward calculation shows that the eigenvectors for α and β satisfy the necessary condition, so $(\alpha, \beta) \in \mathcal{U}$. We can now prove:

Proposition 5.16. *H contains two small elements a and b that generate a free group.*

Proof. Since (α, β) lies in \mathcal{U} , there is some integer m such that α^m and β^m generate a free group. Furthermore since α^m and β^m are infinite order elliptics, there exist integers k and l such that α^{km} and β^{lm} are ϵ_0 -close to the identity.

Since a subgroup of a free group is free, $a = \alpha^{km}$ and $b = \beta^{lm}$ are small elements generating a free group. \square

Our goal now is to use a and b to find a large number of elements near the identity. Let W_i be sets of words in a and b defined inductively as follows. Let $W_0 = \{a, b, a^{-1}, b^{-1}\}$. Let $W_i = \{[x, y] | x, y \in W_{i-1}, x \neq y, x \neq y^{-1}\}$. Let $W = \bigcup W_i$. Let C_i be the image of W_i in the free group $\langle a, b \rangle$. Set $c_i = |C_i|$.

Lemma 5.17. *Distinct words in W have distinct images in the group $\langle a, b \rangle$.*

Proof. The idea is to show that any word in W can be canonically reconstructed from the reduced word representing the same element in the group $\langle a, b \rangle$. In order to describe the reconstruction, it is helpful to describe a step-by-step reduction of a word in W .

This paragraph explains how, given a word $w \in W_i$, words $w_{i-1}, w_{i-2}, \dots, w_0$ are inductively defined, where w_k is a partial reduction of w_{k+1} . Note the indices decrease as the induction proceeds. View w as a word consisting of 4 blocks, each of length 4^{i-1} . By construction no two consecutive blocks are inverses of each other. Set $w_{i-1} = w$. Given w_j , for $j \geq 1$, we construct w_{j-1} by viewing w_j as being made up of blocks of length 4^{j-1} and canceling blocks as follows. Scan w_j from left to right until a block followed by its inverse appears. Cancel these two blocks and then continue scanning from left to right, starting with the block which followed those just cancelled, until another block is followed by its inverse. Cancel these and continue this process until the end of the word is reached. Call the resulting word w_{j-1} .

Conceivably w_{j-1} contains consecutive blocks of length 4^{j-1} which are inverses of each other. The next step is to show that there are not any such pairs. We proceed by induction with the indices decreasing. As noted above w_{i-1} doesn't contain any consecutive blocks of length 4^{i-1} which are inverses of each other. By induction assume that w_j contains no consecutive blocks of length 4^j which are inverses of each other. We show that w_{j-1} contains no consecutive blocks of length 4^{j-1} which are inverses of each other. If there were consecutive blocks of length 4^{j-1} which were inverses of each other, they would have to have come from a sequence in w_j of the form $x_i x_j x_j^{-1} x_i^{-1}$ where x_i and x_j are in W_{j-1} .

Since w_j was gotten by canceling only blocks of length 4^i or longer this sequence must be part of a sequence of the form $x_i^{-1}x_j^{-1}x_ix_jx_j^{-1}x_i^{-1}x_jx_i$, but such a sequence cannot occur since by induction we are assuming that w_j contains no two consecutive blocks which are inverses of each other. Thus w_{j-1} contains no consecutive blocks of length 4^{j-1} which are inverses of each other. In particular this argument shows that w_0 is a reduced word, since it can contain no consecutive blocks of length 4^0 which are inverses of each other.

Let R_0 be the set of all words gotten by completely reducing some word in W . Given a word w_0 in R_0 , we wish to canonically reconstruct the element of W it came from. A word $w_0 \in R_0$ is either in W_0 or it came from reducing a word v_1 with the property that it is made up of blocks of length 4, where each block is an element of W_1 , and no consecutive blocks are inverses of each other. We will say a word v has property (*) if it reduces to w_0 and is made up of blocks of length 4, where each block is an element of W_1 , and no consecutive blocks are inverses of each other.

This paragraph is devoted to showing that there is only one possible word satisfying (*). First we must show that any possible word satisfying (*) must start with the same three letters as v_1 does. The word v_1 must start with some sequence of the form $x_1x_2x_1^{-1}x_2^{-1}$ for some $x_i \in W_0$. If the second block starts with x_2 then when reducing to w_0 the $x_2^{-1}x_2$ cancel but no other cancellation takes place between these two blocks since they aren't allowed to be inverses of each other. Similarly the last and first letter of consecutive blocks may cancel but nothing more, so in particular nothing ever cancels with the first three letters of v_1 . Thus w_0 and v_1 must start with the same three letters. Moreover any word satisfying (*) must start with the same four letters as v_1 . If w_0 and v_1 begin with the same four letters then let $w'_0 = w_0$. Otherwise let w'_0 be the word gotten by inserting the fourth letter of v_1 followed by its inverse into w_0 between its third and fourth letter. We have just shown that the expansion w'_0 of w_0 is independent of the choice of v_1 . The same reasoning shows that any two elements satisfying (*) and beginning with the same $4n$ letters which reduce to w_0 must agree on the first $4(n+1)$ letters. So by induction there is only one w_1 satisfying (*) which reduces to w_0 .

Now say w_i has been reconstructed. Then w_i is either in W_i or it comes from reducing a word w_{i+1} made up of blocks of commutators of elements of W_i such that no two consecutive blocks are inverses of each other. By the same reasoning we used in the case $i = 0$, only one such word exists. Thus by induction we may canonically reconstruct $w \in W$ from w_0 . Thus no two distinct words in W can represent the same element of $\langle a, b \rangle$.

□

We want to count the elements in W and see how the number grows with the word length. Note $c_i = c_{i-1}(c_{i-1} - 2) \geq \frac{1}{2}c_{i-1}^2$.

Observation 5.18. $c_i \geq 2^{2^i}$.

Proof. We will prove that $c_i \geq 2^{2^i+1}$, from which the observation follows. For $n = 1$, we

have $c_1 = 8 = 2^{2^1+1}$. Assume $c_{i-1} \geq 2^{2^{i-1}+1}$. Then,

$$\begin{aligned} c_i &\geq \frac{1}{2}(2^{2^{i-1}+1})^2 \\ &= \frac{1}{2}(2^{2^i+2}) \\ &= 2^{2^i+1}. \end{aligned}$$

□

Theorem 5.19. *For any finitely generated subgroup H of $PSL(2, \mathbf{R})$ which is not discrete or elementary, $|H_{\epsilon_0}^n|$ grows faster than $f(n) = 2^{\frac{\sqrt{n}}{4}}$.*

Proof. Elements in C_i have word length no more than 4^i . So $|H_{\epsilon_0}^{4^i}| \geq 2^{2^i}$. Given n , let j be such that $4^j \leq n < 4^{j+2}$.

$$f(n) = 2^{\frac{\sqrt{n}}{4}} < 2^{2^j} \leq |H_{\epsilon_0}^{4^j}| \leq |H_{\epsilon_0}^n|.$$

□

Theorem 5.20. *G must be discrete.*

Proof. Apply Theorem 5.19 together with Theorem 5.5 taking $\epsilon = \epsilon_0$. □

6. THE KERNEL OF Φ IS QUASI-ISOMETRIC TO \mathbf{R}

To complete our proof we need only show that the kernel of $\Phi : \Gamma \rightarrow G$ is infinite and therefore, by Lemma 5.7, quasi-isometric to \mathbf{R} .

Lemma 6.1. *The kernel of Φ is infinite.*

Proof. Since G is discrete there are only finitely many elements g_u that move x_0 less than any given bounded amount. For any $u \in \Gamma$ such that ϕ_u moves z_0 less than or equal to M_1 horizontally, the corresponding g_u can move x_0 no more than $M_1 + L$. But there are infinitely many such u , since any point in $\mathbf{H}^2 \times \mathbf{R}$ is within M_1 of some orbit point of z_0 . So some element $g \in G$ must be g_u for infinitely many u , say u_1, u_2, u_3, \dots . But then all of the $g_{u^{-1}u_i}$ must be the identity, hence $\ker \Phi$ is infinite. □

Thus we have shown:

Theorem 6.2. *Any group Γ quasi-isometric to $\mathbf{H}^2 \times \mathbf{R}$ is an extension of a finite extension of a cocompact Fuchsian group by a virtually infinite cyclic group. In other words there is an exact sequence*

$$0 \rightarrow A \rightarrow \Gamma \rightarrow G \rightarrow 0$$

where A is virtually infinite cyclic and G is a finite extension of a cocompact Fuchsian group.

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